Le gradient de forme topologique

GDR Applications Nouvelles de l'Optimisation de Formes

Luminy, 7-12 Juillet 2003

Philippe GUILLAUME

The Lagrangian method (Céa)

Shape function

$$j(\alpha) = J(\alpha, u(\alpha))$$

with

$$a(\alpha, u(\alpha), v) = l(\alpha, v), \quad \forall v \in \mathcal{V}.$$

Optimization using a descent method $\Longrightarrow$ differentiate $j(\alpha)$.

Direct method :

$$j'(\alpha) = D_1 J(\alpha, u(\alpha)) + D_2 J(\alpha, u(\alpha)) u'(\alpha).$$

Needs to compute as many vectors $(u'(\alpha))_i$ as the number of parameters α_i.

The *Lagrangian* is defined by

$$\mathcal{L}(\alpha, u, v) = J(\alpha, u) + a(\alpha, u, v) - l(\alpha, v).$$

In particular, for arbitrary v, one has

$$\mathcal{L}(\alpha, u(\alpha), v) = J(\alpha, u(\alpha)) = j(\alpha), \quad \forall \alpha,$$

which allows to compute $j'(\alpha)$ if one differentiates w.r.t. α.

The vector v being fixed, ones has

$$j'(\alpha) = D_1\mathcal{L}(\alpha, u(\alpha), v) + D_2\mathcal{L}(\alpha, u(\alpha), v)u'(\alpha).$$

In particular, choosing $v(\alpha)$ which cancels the second term, i.e. solution to the **adjoint equation**

$$D_2\mathcal{L}(\alpha, u(\alpha), v(\alpha))w = 0, \quad \forall w \in \mathcal{V},$$

which reads also

$$a(\alpha, w, v(\alpha)) = -D_2J(\alpha, u)w, \quad \forall w \in \mathcal{V},$$

then

$$j'(\alpha) = D_1\mathcal{L}(\alpha, u(\alpha), v(\alpha)).$$

Advantage towards the direct method : only two systems to be solved, whatever the number of parameters is.

Generalization

The method generalizes to asymptotic forms like

$$\begin{aligned} a_\varepsilon &= a_0 + \rho(\varepsilon)\delta a + o(\rho(\varepsilon)) \\ l_\varepsilon &= l_0 + \rho(\varepsilon)\delta l + o(\rho(\varepsilon)), \end{aligned}$$

(which can be interpreted as a change of variable $\alpha = \rho(\varepsilon)$), $\varepsilon \geq 0$.
Here again, for $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ and

$$\mathcal{L}_\varepsilon(u, v) = J_\varepsilon(u) + a_\varepsilon(u, v) - l_\varepsilon(v),$$

one has for all v and $\varepsilon \geq 0$

$$j(\varepsilon) = \mathcal{L}_\varepsilon(u_\varepsilon, v).$$

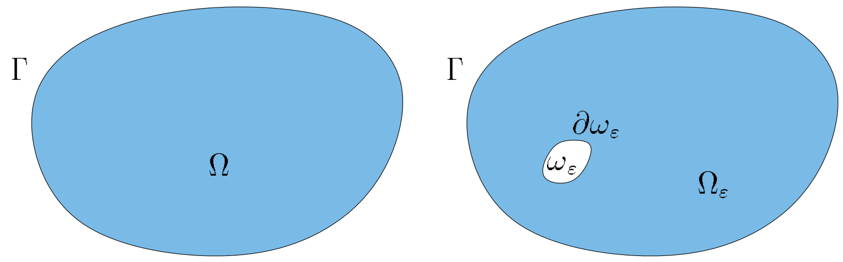
Cancelling the variation w.r.t. u, that is, choosing v_0 solution to

$$D_u\mathcal{L}_0(u_0, v_0)w = 0, \quad \forall w \in \mathcal{V},$$

one obtains, for $\delta\mathcal{L}(u, v) = \delta J(u) + \delta a(u, v) - \delta l(v)$:

$$j(\varepsilon) = j(0) + \rho(\varepsilon)\delta\mathcal{L}(u_0, v_0) + o(\rho(\varepsilon)).$$

Lets see...



The domain before and after perforation, $\omega_\varepsilon = \varepsilon\omega$.

$$\begin{cases} -\Delta u_\varepsilon = f \ \text{ in } \Omega_\varepsilon \\ \quad\ u_\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon. \end{cases}$$

Cost function : supposed to have the form

$$j(\varepsilon) = J(u_\varepsilon).$$

One can consider that $u_\varepsilon \in H_0^1(\Omega)$ with $u_\varepsilon(x) = 0$ if $x \in \omega_\varepsilon$.

The variational formulation is then

$$a_\varepsilon(u_\varepsilon, v) = l(v), \qquad \forall v \in H_0^1(\Omega)$$

with

$$\begin{aligned} a_\varepsilon(u, v) &= \int_{\Omega_\varepsilon} \nabla u.\nabla v \, dx + \int_{\omega_\varepsilon} uv \, dx, \\ l(v) &= \int_\Omega f v \, dx. \end{aligned}$$

The general result tells us that

$$j(\varepsilon) = j(0) + \rho(\varepsilon)\delta a(u_0, v_0) + o(\rho(\varepsilon)),$$

but what is $\delta a(u_0, v_0)$? and $\rho(\varepsilon)$?

$$\begin{aligned} a_\varepsilon(u, v) - a_0(u, v) &= \int_{\omega_\varepsilon} -\nabla u.\nabla v + uv \, dx \\ &\simeq \varepsilon^3 \,\mathrm{mes}(\omega) \left(-\nabla u(0).\nabla v(0) + u(0)v(0)\right) \end{aligned}$$

and by "identification" :

$$\begin{aligned} \delta a(u_0, v_0) &= -\nabla u_0(0).\nabla v_0(0) + u_0(0)v_0(0) \\ \rho(\varepsilon) &= \varepsilon^3 \,\mathrm{mes}(\omega). \end{aligned}$$

However, the correct result is

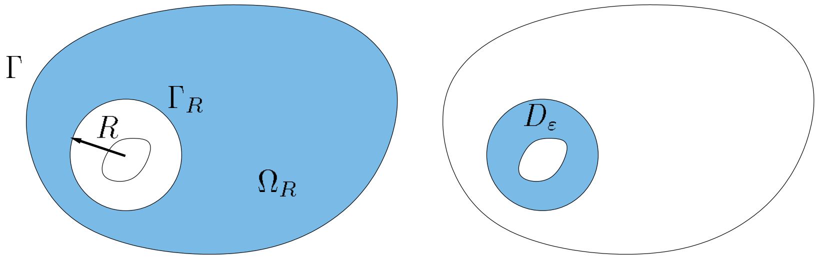
$$j(\varepsilon) = j(0) + \varepsilon A_\omega u_0(0) v_0(0).$$

So ???

Different options for analysis :

- direct study of $a_\varepsilon(u_\varepsilon, v) - a_0(u_0, v)$,
- change functional space to insure continuity of δa,
- ...

Changing the formulation



Decomposition $\Omega_\varepsilon = \Omega_R \cup \Gamma_R \cup D_\varepsilon$.

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla u_\varepsilon.\nabla v \; dx &= \int_{\Omega_R} \nabla u_\varepsilon.\nabla v \; dx + \int_{D_\varepsilon} \nabla u_\varepsilon.\nabla v \; dx \\ &= \int_{\Omega_R} \nabla u_\varepsilon.\nabla v \; dx + \int_{\Gamma_R} \underbrace{\nabla u_\varepsilon.\mathbf{n}}_{T_\varepsilon(u_{\varepsilon|\Gamma_R})} v \; ds\end{aligned}$$

with

$$\begin{aligned}T_\varepsilon : H^{1/2}(\Gamma_R) &\longrightarrow H^{-1/2}(\Gamma_R) \\ \varphi &\longmapsto T_\varepsilon\varphi = \nabla u_\varepsilon^\varphi.\mathbf{n}\end{aligned}$$

where u_ε^φ is solution to

$$\begin{cases} -\Delta u_\varepsilon^\varphi = 0 & \text{in } D_\varepsilon \\ u_\varepsilon^\varphi = \varphi & \text{on } \Gamma_R \\ u_\varepsilon^\varphi = 0 & \text{on } \partial\omega_\varepsilon \end{cases}$$

Functional space (independent of ε) :

$$\mathcal{V}_R = \left\{ u \in H^1(\Omega_R);\ u = 0 \text{ on } \Gamma \right\}$$

Variational formulation :

$$a_\varepsilon(u_\varepsilon, v) = l(v), \qquad \forall v \in \mathcal{V}_R$$

with

$$\begin{aligned} a_\varepsilon(u, v) &= \int_{\Omega_R} \nabla u.\nabla v\, dx + \int_{\Gamma_R} (T_\varepsilon u) v\, ds, \\ l(v) &= \int_{\Omega_R} f v\, dx. \end{aligned}$$

Variation of a_ε

Its analysis reduces to studying $(T_\varepsilon - T_0)\varphi$, that is, $\nabla(u_\varepsilon - u_0).\mathbf{n}$ with

$$\begin{cases} -\Delta u_\varepsilon = 0 \ \text{ in } D_\varepsilon \\ u_\varepsilon = \varphi \ \text{ on } \Gamma_R \\ u_\varepsilon = 0 \ \text{ on } \partial\omega_\varepsilon \end{cases} \qquad \begin{cases} -\Delta u_0 = 0 \ \text{ in } D_0 \\ u_0 = \varphi \ \text{ on } \Gamma_R \end{cases}$$

or $\nabla v_\varepsilon.\mathbf{n}$, with $v_\varepsilon = u_\varepsilon - u_0$ and

$$\begin{cases} -\Delta v_\varepsilon = 0 \ \text{ in } D_\varepsilon \\ v_\varepsilon = 0 \ \text{ on } \Gamma_R \\ v_\varepsilon(x) = -u_0(x) \ \text{ on } \partial\omega_\varepsilon. \end{cases}$$

First approximation (3D, $\omega_\varepsilon = \varepsilon\omega$) : $v_\varepsilon(x) \simeq v_\omega(x/\varepsilon)$ with

$$\begin{cases} -\Delta v_\omega = 0 \ \text{ in } \mathbb{R}^3\backslash\overline{\omega} \\ v_\omega = 0 \ \text{ à l'}\infty \\ v_\omega = -u_0(0) \ \text{ on } \partial\omega. \end{cases}$$

Using integral formulation:

$$v_\omega(y) = \int_{\partial\omega} E(y-z)p(z)\,dz$$

where $E(x) = 1/4\pi r$ is the fundamental solution.
Approximation of $E(y-z)$ for large y and bounded z :

$$E(y-z) = \frac{1}{4\pi||y-z||} = \frac{1}{4\pi||y||} + O\left(\frac{1}{||y||^2}\right).$$

Hence

$$\begin{aligned} v_\omega(y) &= E(y)\int_{\partial\omega} p(z)\,dz + W(y) \\ &= E(y)(-A_\omega u_0(0)) + O\left(\frac{1}{||y||^2}\right) \end{aligned}$$

(which defines constant A_ω), and

$$v_\varepsilon(x) \simeq v_\omega(x/\varepsilon) = -\varepsilon E(x) A_\omega u_0(0) + \underbrace{W(x/\varepsilon)}_{O(\varepsilon^2) \text{ far from } \partial\omega_\varepsilon}.$$

On the boundary of D_ε, the difference $v_\varepsilon(x) - v_\omega(x/\varepsilon)$ has value

$$\begin{cases} -u_0(x) + u_0(0) &= O(\varepsilon) \text{ on } \partial\omega_\varepsilon \\ \varepsilon\, A_\omega u_0(0) E(x) + O(\varepsilon^2) &= O(\varepsilon) \text{ on } \Gamma_R. \end{cases}$$

We need the following tool:

Lemma 0.1 *If*

$$\begin{cases} -\Delta w_\varepsilon = 0 \text{ in } D_\varepsilon \\ \quad w_\varepsilon = \varphi \text{ on } \Gamma_R \\ \quad w_\varepsilon = \psi \text{ on } \partial\omega_\varepsilon, \end{cases}$$

then

$$|w_\varepsilon|_{1,C(R/2,R)} \leq c \left(\|\varphi\|_{1/2,\Gamma_R} + \varepsilon \, \|\psi(\varepsilon y)\|_{1/2,\partial\omega} \right).$$

However, something is missing: the approximation is of order ε, but the remainder also... $\rightarrow$ we need to add a term with boundary condition $\varepsilon A_\omega u_0(0) E(x)$ on Γ_R.

Second approximation (suppress the term of order ε) *:*

$$v_\varepsilon(x) = -\varepsilon A_\omega u_0(0)\,[E(x) - Q(x)] + r(x)$$

where

$$\begin{cases} -\Delta Q &= 0 \text{ in } D_0 \\ \quad Q &= E \text{ on } \Gamma_R, \end{cases}$$

(here $Q =$ constant), the remainder being

$$r(x) \;=\; \underbrace{v_\varepsilon(x) - v_\omega(x/\varepsilon) - \varepsilon A_\omega u_0(0)Q(x)}_{r_1(x)} + W(x/\varepsilon)$$

with

$$\begin{cases} -\Delta r_1 = 0 & \text{in } D_\varepsilon \\ r_1 = -W(x/\varepsilon) & = O(\varepsilon^2) \text{ for norm } \|.\|_{1/2,\Gamma_R} \text{ on } \Gamma_R \\ r_1 = -u_0(x) + u_0(0) - \varepsilon A_\omega u_0(0)Q(x) & = O(\varepsilon) \text{ for } \|.(\varepsilon.)\|_{1/2,\partial\omega} \text{ on } \partial\omega_\varepsilon \end{cases}$$

Consequence:

$$\begin{aligned}(T_\varepsilon - T_0)\varphi &= \nabla(u_\varepsilon - u_0).\mathbf{n} \\ &= -\varepsilon A_\omega u_0(0)\nabla\left[E(x) - Q(x)\right].\mathbf{n}+O(\varepsilon^2)\end{aligned}$$

and (substituting $u_{|\Gamma_R}$ for φ):

$$\begin{aligned}a_\varepsilon(u,v) - a_0(u,v) &= \int_{\Gamma_R}(T_\varepsilon u - T_0 u)\, v\, ds \\ &= \int_{\Gamma_R}(\nabla(u_\varepsilon - u_0).\mathbf{n})\, v\, ds \\ &= -\varepsilon A_\omega u_0(0)\int_{\Gamma_R}\nabla\left[E(x) - Q(x)\right].\mathbf{n}\, v\, ds+O(\varepsilon^2) \\ &= \varepsilon A_\omega u_0(0)\frac{1}{4\pi R^2}\int_{\Gamma_R} v\, ds+O(\varepsilon^2)\ \text{(here } \nabla Q = 0\text{)}.\end{aligned}$$

Moreover, if $v = v_0$ with $\Delta v_0 = 0$, then

$$j(\varepsilon) - j(0) = \varepsilon A_\omega u_0(0)v_0(0)+O(\varepsilon^2)$$

where $A_\omega = 4\pi$ if $\omega = B(0,1)$.

Examples

Example 1 :

$$J_\varepsilon(u) = \int_{\Omega_\varepsilon} g(x, u(x))\, dx.$$

Adjoint state:

$$-\Delta v_\Omega = -D_2 g(x, u_\Omega)$$

Gradient: it is the same as before,

$$j(\varepsilon) = j(0) + \varepsilon\, A_\omega u_\Omega(x_0) v_\Omega(x_0) + o(\varepsilon).$$

Example 2 :

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} B(x)\nabla(u - u_{\mathrm{d}}).\nabla(u - u_{\mathrm{d}})\,dx.$$

Adjoint state:

$$-\Delta v_\Omega = \mathrm{div}\,(B(\nabla u_\Omega - \nabla u_{\mathrm{d}}))$$

Gradient : a second term coming from δJ appears,

$$j(\varepsilon) = j(0) + \varepsilon A_\omega u_\Omega(x_0)v_\Omega(x_0) + \frac{1}{2}\int_{D_\varepsilon} B\nabla_x v_\omega(x/\varepsilon).\nabla_x v_\omega(x/\varepsilon)\,dx + o(\varepsilon)$$

where

$$\begin{aligned}\frac{1}{2}\int_{D_\varepsilon} B(x)\nabla_x v_\omega(x/\varepsilon).\nabla_x v_\omega(x/\varepsilon)\,dx &= \frac{\varepsilon}{2}\int_{D_\varepsilon/\varepsilon} B(\varepsilon y)\nabla v_\omega(y).\nabla v_\omega(y)\,dy \\ &= \frac{\varepsilon}{2}\int_{\mathbb{R}^3\setminus\overline{\omega}} B(x_0)\nabla v_\omega(y).\nabla v_\omega(y)\,dy.\end{aligned}$$

Optimization algorithm

- Ω the current domain,
- compute the solution u and the adjoint state v,
- compute $\delta j(x) = \mathcal{J}(u, v, \nabla u, \nabla v, x)$,
- find x_0 which minimizes δj,
- if $\delta j(x_0) < 0$ (or mes$(\Omega) < m$, or ...)
  - linear search : choose $t > 0$ such that

$$\begin{aligned} j(\Omega_t) &< j(\Omega) \\ \Omega_t &= \{x \in \Omega;\ \delta j(x) - \delta j(x_0) > t\} \end{aligned}$$

  - $\Omega \leftarrow \Omega_t$ and back to beginning,
- else: stop.

Remark : for Poisson's equation, the optimality condition $\delta j(x) > 0$ reads

$$u(x)v(x) > 0, \ \forall x \in \Omega.$$

This condition was found by Buttazzo and Dal Maso for the relaxed problem

$$\begin{cases} -\Delta u_\mu + \mu u_\mu = f & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega \end{cases}$$

where μ is a positive measure on Ω which cancels on sets of capacity zero.

Local minimum (example)

Case 2D Neumann with $J =$ energy. In this case, $v = -u$.

- In the classical shape optimization context,

$$
\begin{aligned}
Dj(\Omega)V &= -\int_\Gamma |\nabla u|^2 V.\mathbf{n}\, ds \\
&= -|\nabla u|^2 \times \text{ area variation.}
\end{aligned}
$$

- In the topological context,

$$
\begin{aligned}
j(\varepsilon) - j(0) &\simeq 2\pi\varepsilon^2 |\nabla u|^2 \\
&= 2\,|\nabla u|^2 \times \text{ area variation.}
\end{aligned}
$$

In particular, under the area constraint

$$\mathrm{mes}(\Omega) = \text{constant},$$

if

$$2 \min_{\Omega} |\nabla u|^2 > \max_{\Gamma} |\nabla u|^2,$$

then adding the two variations (top. and boundary, with constant surface) yields

$$\begin{aligned}
\delta(j) &= -\int_{\Gamma} |\nabla u|^2 V.\mathbf{n}\, ds + 2\pi\varepsilon^2 |\nabla u|^2 + o(\varepsilon^2) \\
&\geq \Big(-\max_{\Gamma} |\nabla u|^2 + 2\min_{\Omega} |\nabla u|^2 + o(1) \Big) \times \text{area variation} \\
&> 0.
\end{aligned}$$

A classical shape can be a local minimum. Same for elasticity: full stress.

Numerical example : Poisson

$$j(\Omega) = \int_{B(0,1)} (u_\Omega - u_d)^2 \, dx$$

where $u_\Omega \in H_0^1(\Omega)$ is the solution to

$$-\Delta u_\Omega = 1 \text{ in } \Omega \subset \overline{B}(0,1) \subset \mathbb{R}^2,$$

extended by 0 on $\overline{B}(0,1) \backslash \overline{\Omega}$, and

$$u_d(x) = \begin{cases} \frac{r_1^2 - r^2}{4} + a & \text{if } \; r \leq r_1 \\ a & \text{if } \; r_1 \leq r \leq 1 \end{cases}$$

with $0 < a < 3/16$, and r_1 is the first minimum of

$$q(r_1) = 2\pi \int_{r_1}^{1} \left(\left(a - \frac{1 - r_1^2}{4} \right) \frac{\log r}{\log r_1} + \frac{1 - r^2}{4} - a \right)^2 r \, dr.$$

There is no classical solution.

Relaxed formulation:

$$\min_{\mu} \int_{B(0,1)} (u_\mu - u_d)^2 \, dx$$

where $u_\mu \in H_0^1(B(0,1))$ is the solution to

$$-\Delta u_\mu + \mu u_\mu = 1 \text{ in } B(0,1)$$

The solution is known:

$$\mu \;=\; \gamma H^1_{\partial B(0,r_1)}, \quad \gamma = \frac{4a - 1 + r_1^2}{4ar_1 \log r_1}$$

$$u_\mu(x) \;=\; \begin{cases} u_d(x) & \text{si } r \le r_1 \\ \left(a - \frac{1-r_1^2}{4}\right) \frac{\log r}{\log r_1} + \frac{1-r^2}{4} & \text{si } r_1 \le r \le 1 \end{cases}$$

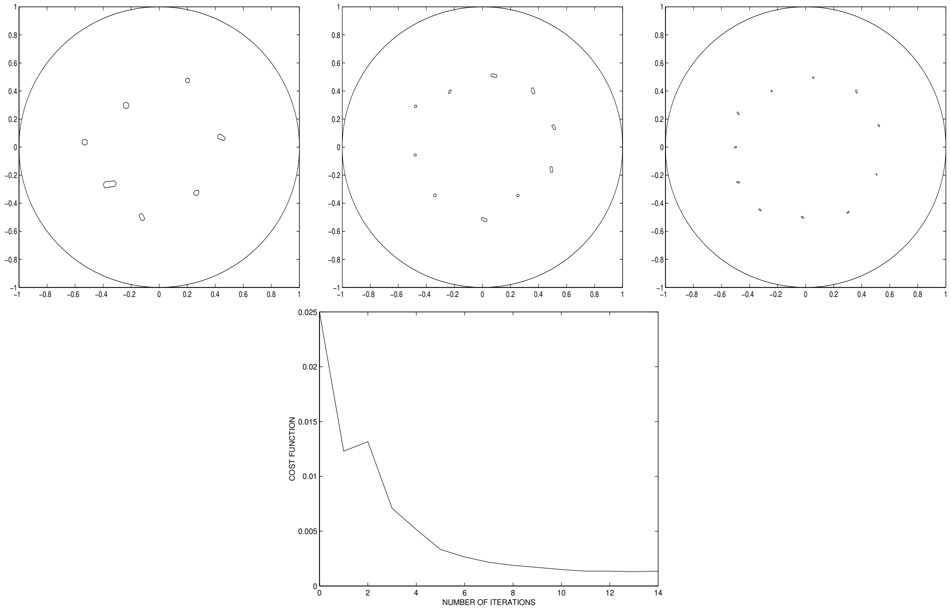
Minimum :

$$\int_{B(0,1)} (u_\mu - u_d)^2 \, dx = q(r_1) \simeq 1.12\, 10^{-3}.$$

Minimizing sequence:

$$\Omega_N = B(0,1) \setminus \bigcup_{k=1}^{N} \left\{ z \in \mathbb{C}\,;\, \left| z - \mathrm{e}^{2i\pi k/N} \right| \leq \mathrm{e}^{-N/\gamma r_1} \right\}.$$

Via topological optimization, one finds for $h = .05$, $.025$ and $.0125$ (cost function for $h = 0.025$) :

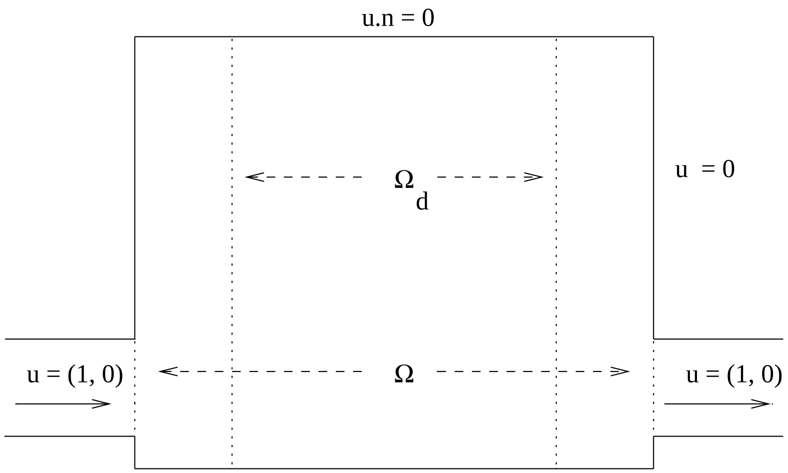


Numerical example: Stokes

A tank filled with an incompressible fluid:

$$-\nu \Delta u_\Omega + \nabla p_\Omega = 0 \text{ in } \Omega = [0, 1.4] \times [-1.2, 1.2].$$

.



We can insert some obstacles in order to approximate a target flow u_{d}.
The cost function is

$$J(u) = \int_{\Omega_d} ||u - u_{\mathrm{d}}||^2 \, dx$$

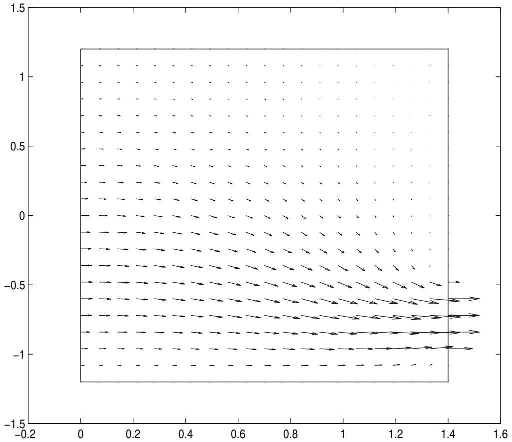
$$u_{\mathrm{d}} = \left(\frac{y+1.2}{2.4^2}, 0 \right), \quad \Omega_{\mathrm{d}} = \{(x,y) \in \Omega; \ |x| < 0.8\}.$$

Adjoint equation:

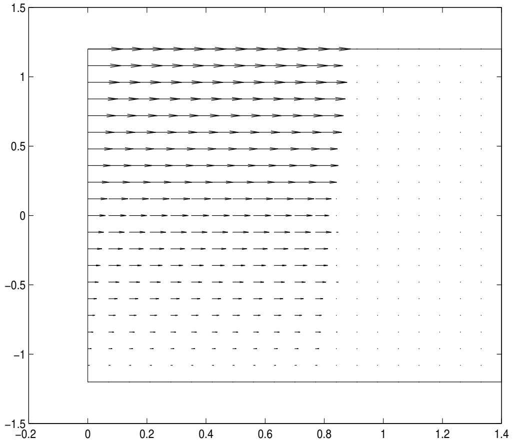
$$\begin{cases} -\nu \Delta v_\Omega + \nabla q_\Omega = -2(u_\Omega - u_{\mathrm{d}}) \text{ in } \Omega, \\ v_\Omega = 0 \text{ on } \Gamma, \end{cases}$$

Gradient:

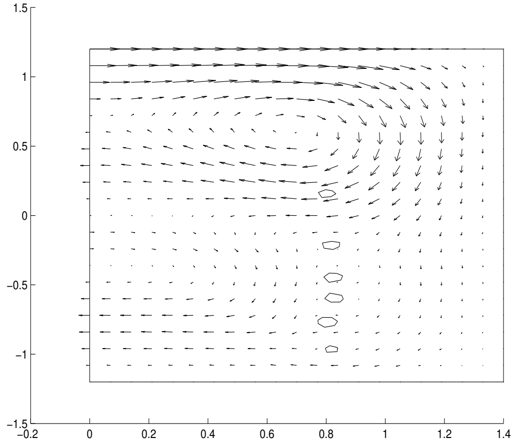
$$j(\varepsilon) = j(0) - \frac{4\pi\nu}{\log\varepsilon} u_\Omega(x_0).v_\Omega(x_0) + o\left(\frac{-1}{\log\varepsilon}\right).$$



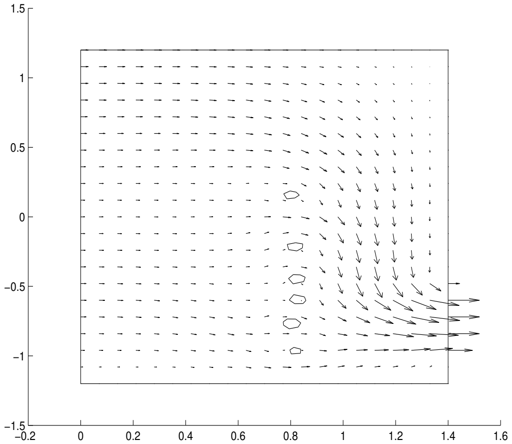
(a) initial flow



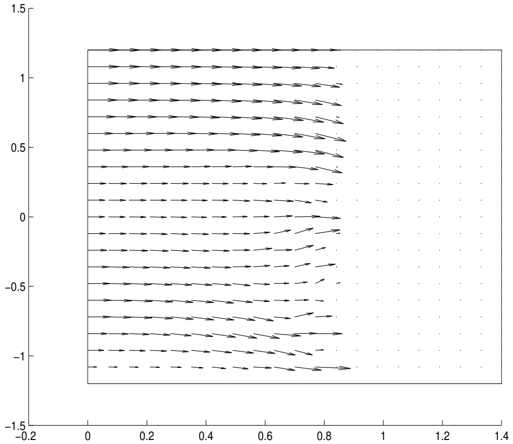
(b) objective flow



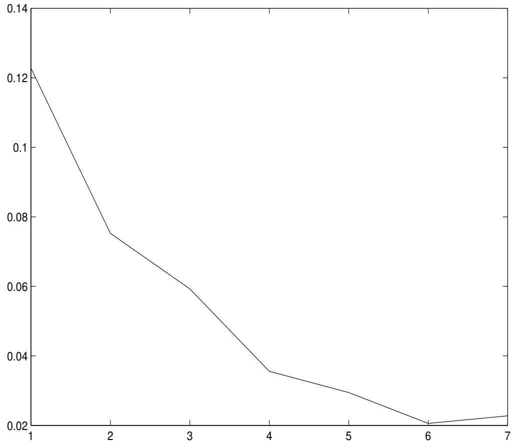
(e) adjoint flow



(c) obtained flow on Ω



(d) obtained flow on Ω_d



(f) cost function